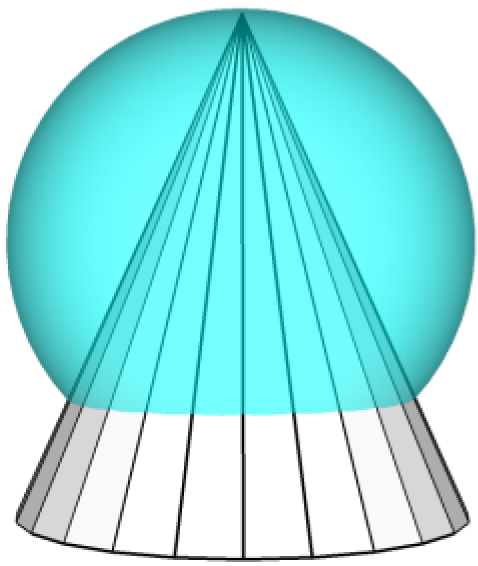
Timoshenko Particles on a Cone

A fellow traveler through the waste lands of Greek geometry called my attention to this problem from Stephen Timoshenko's "Advanced Dynamics" text. Timoshenko described it thusly:

Under the influence of gravity and without resistance, particles start simultaneously from rest at time 0 and slide along variously inclined straight grooves cut in the surface of a cone with an inclined axis.

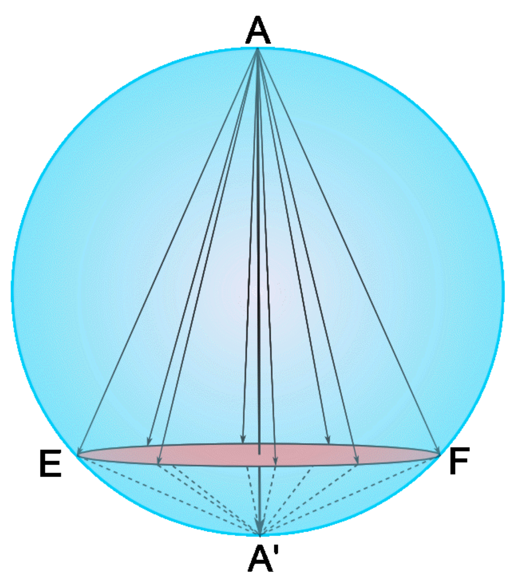
and the reader is asked to:

Prove that after any lapse of time t, the particles lie on the surface of a sphere of diameter $d = \frac{1}{2}gt^2$.



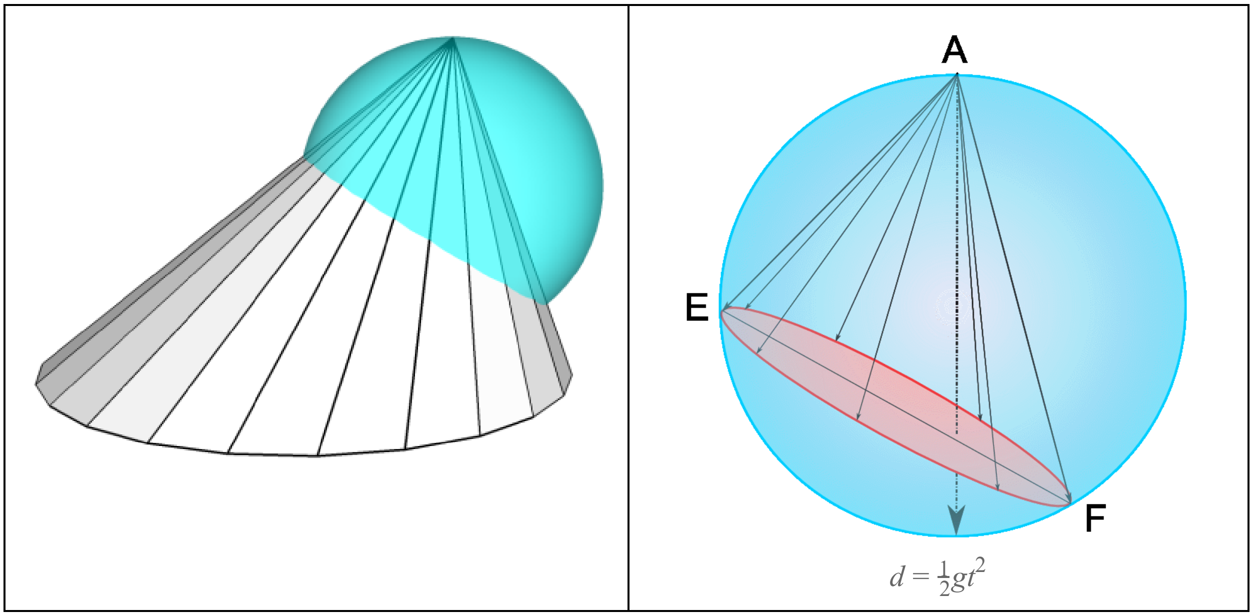
What's asked to be proven is also true when the cone is right and the proof much easier to visualize. Consider the right cone in the figure to the right where a sphere of diameter $d = \frac{1}{2}gt^2$ has been placed on the cone in such a manner that a diameter of the sphere is aligned collinearly with the cone axis and has its upper end at the cone apex.

Planes parallel to the base of circular cone are known to intersect the cone in circle. Similarly a plane always intersects a sphere in a circle that is perpendicular to a diameter of the sphere. In this scenario the sphere and cone intersect on a plane parallel to the cone base producing concurrent circles. The question remains, however, where are the particles?



The portion of the cone generators between the apex and their intersection with the sphere are now also chords of the sphere and are of equal lengths due to the symmetry of the cone. Each chord is one side of a right angle inscribed in a semicircle of the sphere. As such it is one side of a right triangle that has the sphere's diameter as its hypotenuse and which effectively decomposes the distance along the diameter into orthogonal components. The component along a chord being the distance travelled by a particle along the corresponding generator segment in time t. All chords end on the intersection of the cone with the sphere. Hence at time t the particles will lie on the circle of that intersection.

When a sphere is placed on an oblique cone with a diameter aligned along the cone altitude rather than the cone axis, so as to correctly reflect the free fall direction, it is obvious that the intersection of the sphere with the cone does not occur on a plane parallel to the base.

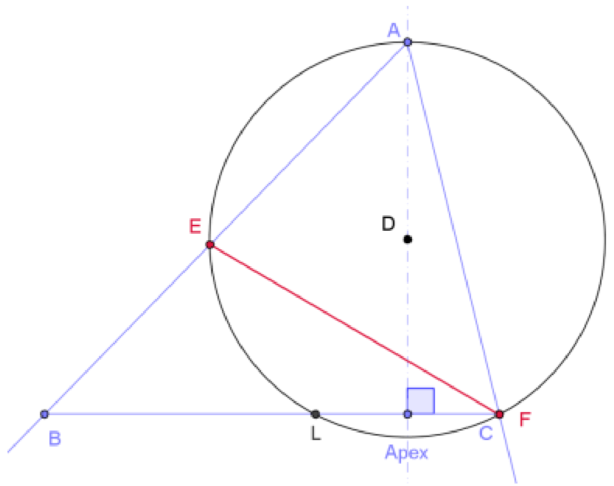


Instead the intersection occurs on a “subcontrary” or anti-parallel plane that exists in oblique cones. Oblique circular cones are in reality elliptical cones cut at such an angle as to produce a circular base. When it is recognized that, as before, the intersection occurs on a plane, the plane can be realigned in the distance vector sphere so that the chords again intersect the sphere at its intersection with the cone. When this is done the chord lengths will then be the distance travelled along the corresponding generator segment and, as before, show that the particles lie on the coincident circles of the intersection.

Oblique cones seem to have become extinct in today’s geometry studies suggesting than an elaboration on some of their properties is appropriate here. Of particular interest are those that involve the upright axial triangle formed by the base diameter on which the cone altitude falls and the two generator lines from the ends of the diameter to the apex. The plane of the upright triangle divides the cone into mirror images and hence indirectly congruent halves.

In addition to the cone altitude, the cone axis - line from center of the base to the apex - lies on this plane as does the rotational symmetry axis and the symmedian - line from the apex to the base through the centers of the circular subcontrary sections. The properties of the subcontrary plane were known before him but Apollonius is credited with defining it as being a plane perpendicular to the upright axial plane “that cuts the cone in such a way that the portion of the axial triangle cut off by the plane is indirectly similar to the whole axial triangle.”

Though often spoken of as if there is a single unique subcontrary plane, the subcontraries are a collection of parallel planes. If a particular plane is known to be a subcontrary then any plane intersecting with the cone parallel to it is also a subcontrary plane. This drawing shows the intersection of the sphere with the upright triangle of the cone. In this instance it is easier to prove that EF is a subcontrary if endpoint F is concurrent with the cone vertex C.



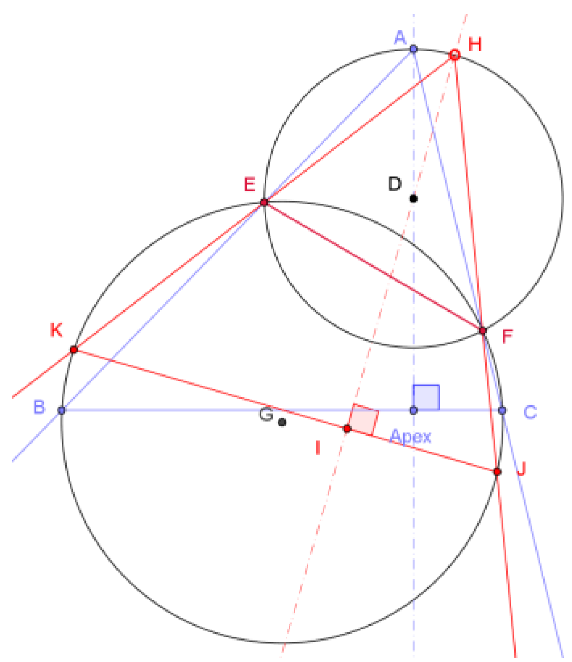
By dragging the center D along the altitude the circle can be expanded until point F is concurrent with the cone vertex C while keeping EF parallel to its previous orientation.

The altitude $AApex$ is perpendicular to and bisects the chord LC making $arcAEL = arcAC$ and in turn the included angle $\angle AEC = \angle ACB$. $\angle EAC$ is common to both $\triangle EAC$ and $\triangle ABC$. Having two angles congruent, the two triangles are similar making the cone and the part cut off by the plane indirectly similar. EF thus is shown to be a subcontrary plane.

EF being antiparallel to the base BC is sufficient to insure that the four points $B, E, F,$ and J lie on the circle which was drawn after locating its center G.

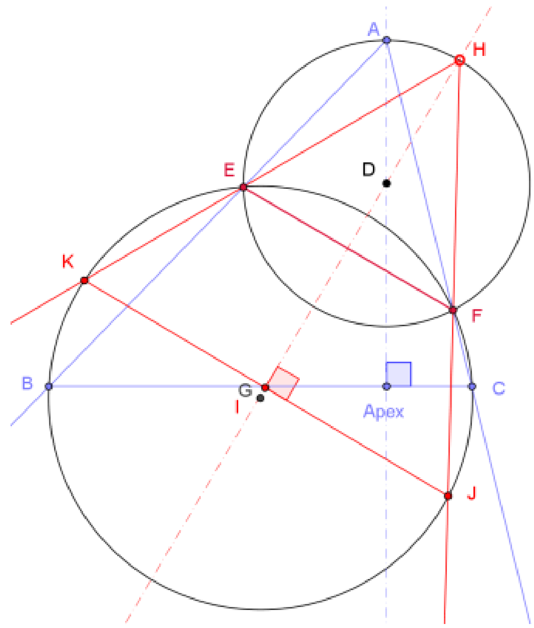
In addition to the original cone ABC that is fixed, a second cone is added with its apex H constrained to move on the upper circle. Rays are drawn from H through E and F and their second intersection with the lower circle at K and J which are constrained to move on the lower circle.

The altitude of the second cone passes through the center of the upper circle and is perpendicular to the base KJ where it intersects it at I. I is free to move along KJ and its extension.



When H is moved to A the red and blue cones are congruent and coincident. Angle EHF intercepts a constant $arcEF$ in the upper circle as H is moved around the circle. But the sides or the triangle HKJ are also secants to the lower circle. Thus the measure of angle EHF is also equal one half the difference between the measure of arcs intercepted by KJ and EF in the lower circle. Since $arcEF$ is again constant, $arcKJ$ is also constant and in turn the base of the red cone, $chordKJ$, is constant.

The real significance here, however, is that the base is always oriented such that EF remains the subcontrary plane of cone HKJ. As apex H is moved the cone adjusts so that the intersection between the cone and the plane is always a circle. Further, because a diameter of the sphere is always aligned with the altitude of the cone, the generator lines of the cone are still distance vectors of the circle (sphere) and properly predict the particle locations at time t.



When H is positioned so that the altitude is perpendicular to the plane through EF, as shown here, HKJ is a right cone. In a right cone, the base plane, the cone cross section plane and subcontrary plane are the same plane. And, as shown previously the particles lie on the coincident circles of the intersection between the sphere and the cone.

The GeoGebra model used for the drawings can be accessed via this link [Timoshenko Circles](). It can be manipulated by dragging the red apex H or the blue point labeled Apex. In both cases the diameter of the sphere (circle with center D) is a constant $d = \frac{1}{2}gt^2$. It can be changed by dragging D.

Dragging apex H scrolls through cones keeping the diameter of the intersection subcontrary circle and the cone base diameter constant while varying the cone altitude, shape and apex angle.

Dragging the blue point labelled $Apex$ scrolls through cones keeping the cone base and altitude constant while changing the cone shape and apex angle and varying the diameter of the intersection subcontrary circle and its orientation as need to keep it antiparallel to the cone base.

My thanks to Ty of ty@tyharness.co.uk for bringing this problem to my attention.

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